Optimal Portfolio under Fractional Stochastic Environment

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Jim Gatheral's 60th Birthday Conference
NYU October 13-15, 2017

Portfolio Optimization: Merton's Problem

An investor manages her portfolio by investing in a riskless asset B_t and in a risky asset S_t (single asset for simplicity)

$$\begin{cases} \mathrm{d}B_t = rB_t\,\mathrm{d}t \\ \mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t \end{cases}$$

- π_t – amount of wealth invested in the risky asset at time t
- X_t^π – the wealth process associated to the strategy π

$$\begin{aligned} \mathrm{d}X_t^\pi &= \frac{\pi_t}{S_t}\,\mathrm{d}S_t + \frac{X_t^\pi - \pi_t}{B_t}\,\mathrm{d}B_t \quad \text{(self-financing)} \\ &= (rX_t^\pi + \pi_t(\mu - r))\,\mathrm{d}t + \pi_t\sigma\,\mathrm{d}W_t \end{aligned}$$

Objective:

$$M(t,x;\lambda) := \sup_{\pi\in\mathcal{A}(x,t)} \mathbb{E}\left[U(X_T^\pi)|X_t^\pi = x\right], \quad \text{Sharpe ratio: } \lambda = \frac{\mu}{\sigma}$$

where $\mathcal{A}(x)$ contains all admissible π and $U(x)$ is a utility function on $\mathbb{R}^+$

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Stochastic Volatility

- In Merton's work, μ and σ are constant, complete market
- Empirical studies reveals that σ exhibits "random" variation
- Implied volatility skew or smile
- Stochastic volatility model: $\mu(Y_t), \sigma(Y_t) \rightarrow$ incomplete market
- Rough Fractional Stochastic volatility:
  - Gatheral, Jaisson and Rosenbaum '14 (*"volatility is rough"*)
  - Jaisson, Rosenbaum '16 (*"from Hawkes processes to fractional diffusions"*)
  - Omar, Masaaki and Rosenbaum '16 (*"leveraged rough volatility"*)

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Nonlinear + Non-Markovian $\rightarrow$ HJB PDE not avaialbe

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Related Literature

- Option Pricing + Markovian modeling:
  Fouque, Papanicolaou, Sircar and Solna '11 (CUP)
- Portfolio Optimization + Markovian modeling:
  Fouque, Sircar and Zariphopoulou '13 (MF)
  Fouque and Hu '16 (SICON)
- Option Pricing + Non-Markovian modeling:
  Garnier and Solna '15 (SIFIN), '16 (MF)
- Portfolio Optimization + Non-Markovian modeling:
  Fouque and Hu arXiv:1703.06969 (slow factor)
  Fouque and Hu arXiv:1706.03139 (fast factor)

A General Non-Markovian Model

Dynamics of the risky asset S_t

$$\begin{cases} \mathrm{d}S_t = S_t \left[\mu(Y_t)\,\mathrm{d}t + \sigma(Y_t)\,\mathrm{d}W_t \right], \\ Y_t\text{: a general stochastic process, } \mathcal{G}_t := \sigma\left\{ \left(W^Y\right)_{0 \le u \le t} \right\}\text{-adapted,} \end{cases}$$

with
$$\mathrm{d}\left\langle W, W^Y \right\rangle_t = \rho\,\mathrm{d}t.$$

Dynamics of the wealth process X_t (assume $r = 0$ for simplicity):

$$\mathrm{d}X_t^\pi = \pi_t \mu(Y_t)\,\mathrm{d}t + \pi_t \sigma(Y_t)\,\mathrm{d}W_t$$

Define the value process V_t by

$$V_t := \sup_{\pi \in \mathcal{A}_t} \mathbb{E}\left[U(X_T^\pi) \middle| \mathcal{F}_t \right]$$

where $U(x)$ is of power type $U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0.$

Proposition: Martingale Distortion Transformation[1]

- The value process V_t is given by

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma}\left[\widetilde{\mathbb{E}}\left(e^{\frac{1-\gamma}{2q\gamma}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s}\,\middle|\,\mathcal{G}_t\right)\right]^q, \quad \lambda(y) = \frac{\mu(y)}{\sigma(y)}$$

where under $\widetilde{\mathbb{P}}$, $\widetilde{W}_t^Y := W_t^Y + \int_0^t a_s\,\mathrm{d}s$ is a BM.

- The optimal strategy π^* is

$$\pi_t^* = \left[\frac{\lambda(Y_t)}{\gamma\sigma(Y_t)} + \frac{\rho q \xi_t}{\gamma\sigma(Y_t)}\right] X_t$$

where ξ_t is given by the martingale representation $\mathrm{d}M_t = M_t \xi_t\,\mathrm{d}\widetilde{W}_t^Y$ and M_t is

$$M_t = \widetilde{\mathbb{E}}\left[e^{\frac{1-\gamma}{2q\gamma}\int_0^T \lambda^2(Y_s)\,\mathrm{d}s}\,\middle|\,\mathcal{G}_t\right]$$

[1] Tehranchi '04: different utility function, proof and assumptions

Remarks

- only works for one factor models
- assumptions: integrability conditions of ξ_t, X_t^π and π_t
- $\gamma = 1 \to$ case of log utility, can be treated separately
- degenerate case $\lambda(y) = \lambda_0$, M_t is a constant martingale, $\xi_t = 0$

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda_0^2(T-t)}, \quad \pi_t^* = \frac{\lambda_0}{\gamma\sigma(Y_t)} X_t.$$

- uncorrelated case $\rho = 0$, the problem is "linear" since $q = 1$

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Sketch of Proof (Verification)

- V_t is a supermartingale for any admissible control π
- V_t is a true martingale following π^*
- π^* is admissible

Define $\alpha_t = \pi_t / X_t$, then

$$\mathrm{d}V_t = V_t D_t(\alpha_t)\,\mathrm{d}t + \mathrm{d}\,\text{Martingale}$$

with the drift factor $D_t(\alpha_t)$

$$D_t(\alpha_t) := \alpha_t \mu - \frac{\gamma}{2}\alpha_t^2\sigma^2 - \frac{\lambda^2}{2\gamma} + \frac{q}{1-\gamma}a_t\xi_t + \frac{q(q-1)}{2(1-\gamma)}\xi_t^2 + \rho q \alpha_t \sigma \xi_t$$

maximize $D_t \Rightarrow \alpha_t^*$ and $D_t(\alpha_t^*) = 0$ with the right choice of a_t and q:

$$a_t = -\rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Y_t), \quad q = \frac{\gamma}{\gamma + (1-\gamma)\rho^2}.$$

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Relation to the Distortion Transformation [2]

In the Markovian setup, Y_t is a diffusion process

$$\mathrm{d}Y_t = k(Y_t)\,\mathrm{d}t + h(Y_t)\,\mathrm{d}W_t^Y,$$

$V(t,x,y)$ solves a HJB equation and the distortion transformation is

$$V(t,x,y) = \frac{x^{1-\gamma}}{1-\gamma}\Psi(t,y)^q$$

where $\Psi(t,y)$ solves the linear PDE

$$\Psi_t + \left(\frac{1}{2}h^2(y)\partial_{yy} + k(y)\partial_y + \rho\frac{1-\gamma}{\gamma}\lambda(y)h(y)\partial_y\right)\Psi + \frac{1-\gamma}{2q\gamma}\lambda^2(y)\Psi = 0,$$

and has the probabilistic representation

$$\Psi(t,y) = \widetilde{\mathbb{E}}\left[e^{\frac{1-\gamma}{2q\gamma}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s}\,\middle|\,Y_t = y\right].$$

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Multiple Assets Modeling

Consider the following model of S_t^1, S_t^2, …, S_t^n

$$\mathrm{d}S_t^i = \mu^i(Y_t^i)S_t^i\,\mathrm{d}t + \sum_{j=1}^{k} \sigma_{ij}(Y_t^i)S_t^i\,\mathrm{d}W_t^j, \quad i = 1, 2, \dots n.$$

Each S_t^i is driven by a stochastic factor Y_t^i, but all factors Y_t^i are adapted to the same single Brownian motion W_t^Y with the correlation structure:

$$\mathrm{d}\left\langle W^i, W^j \right\rangle_t = 0, \quad \mathrm{d}\left\langle W^i, W^Y \right\rangle_t = \rho\,\mathrm{d}t, \quad \forall\, i, j = 1, 2, \dots, n.$$

Martingale Distortion Transformation with Multiple Assets

Then, the portfolio value V_t can be expressed as

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma}\left[\widetilde{\mathbb{E}}\left(e^{\frac{1-\gamma}{2q\gamma}\int_t^T \mu(\mathbf{Y}_s)^\dagger \Sigma(\mathbf{Y}_s)^{-1}\mu(\mathbf{Y}_s)\,\mathrm{d}s}\,\middle|\,\mathcal{G}_t\right)\right]^q,$$

the constant q is chosen to be:

$$q = \frac{\gamma}{\gamma + (1-\gamma)\rho^2 \mathbb{1}_k^\dagger \sigma(\mathbf{Y}_t)^\dagger \Sigma^{-1}(\mathbf{Y}_t)\sigma(\mathbf{Y}_t)\mathbb{1}_k},$$

and $\mathbb{1}_k$ is a k-vector of ones. The optimal control π^* is given by

$$\pi_t^* = \left[\frac{\Sigma(\mathbf{Y}_t)^{-1}\mu(\mathbf{Y}_t)}{\gamma} + \frac{\rho q \xi_t \Sigma(\mathbf{Y}_t)^{-1}\sigma(\mathbf{Y}_t)\mathbb{1}_k}{\gamma}\right] X_t.$$

Fractional Processes

A fractional Brownian motion $W_t^{(H)}$, $H \in (0,1)$

- a continuous Gaussian process
- zero mean
- $\mathbb{E}\left[W_t^{(H)} W_s^{(H)}\right] = \frac{\sigma_H^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$
- $H < 1/2$: short-range correlation; $H > 1/2$: long-range correlation

A fractional Ornstein–Uhlenbeck process solves

$$\mathrm{d}Z_t^H = -aZ_t^H\,\mathrm{d}t + \mathrm{d}W_t^{(H)}$$

- stationary solution $Z_t^H = \int_{-\infty}^t e^{-a(t-s)}\,\mathrm{d}W_s^{(H)} = \int_{-\infty}^t \mathcal{K}(t-s)\,\mathrm{d}W_s^Z$
- Gaussian process with zero mean and constant variance
- $\mathcal{K}$ is non-negative, $\mathcal{K} \in L^2$

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Merton Problem under Slowly Varying Fractional SV

Consider a rescaled stationary fOU process $Z_t^{\delta,H}$

$$\begin{cases} \mathrm{d}S_t = S_t \left[\mu(Z_t^{\delta,H})\,\mathrm{d}t + \sigma(Z_t^{\delta,H})\,\mathrm{d}W_t\right], \\ Z_t^{\delta,H} = \int_{-\infty}^t \mathcal{K}^\delta(t-s)\,\mathrm{d}W_s^Z, \quad \mathcal{K}^\delta(t) = \sqrt{\delta}\mathcal{K}(\delta t), \end{cases} \quad \mathrm{d}\left\langle W, W^Z \right\rangle_t = \rho\,\mathrm{d}t.$$

Our study gives, for all $H \in (0,1)$:

- The value process $V_t^\delta := \operatorname{ess\,sup}_{\pi \in \mathcal{A}_t^\delta} \mathbb{E}\left[U(X_T^\pi) \middle| \mathcal{F}_t\right]$
- The corresponding optimal strategy π^*
- First order approximations to V_t^δ and π^*
- A practical strategy to generate this approximated value process

Apply the martingale distortion transformation with $Y_t = Z_t^{\delta,H}$

$$V_t^\delta = \frac{X_t^{1-\gamma}}{1-\gamma}\left[\widetilde{\mathbb{E}}\left(e^{\frac{1-\gamma}{2q\gamma}\int_t^T \lambda^2(Z_s^{\delta,H})\,\mathrm{d}s}\middle| \mathcal{G}_t\right)\right]^q,$$

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Approximation to the Value Process

Theorem (Fouque-Hu '17)

For fixed $t \in [0, T)$, $X_t = x$ and the observed value $Z_0^{\delta,H}$, the value process V_t^δ takes the form

$$V_t^\delta = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda^2(Z_0^{\delta,H})(T-t)} + \frac{X_t^{1-\gamma}}{\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda^2(Z_0^{\delta,H})(T-t)} \lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H}) \phi_t^\delta$$

$$+ \delta^H \rho \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda^2(Z_0^{\delta,H})(T-t)} \lambda^2(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H}) \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{(T-t)^{H+\frac{3}{2}}}{\Gamma(H+\frac{5}{2})}$$

$$+ \mathcal{O}(\delta^{2H}),$$

where ϕ_t^δ is the random component of order δ^H

$$\phi_t^\delta = \mathbb{E}\left[\int_t^T \left(Z_s^{\delta,H} - Z_0^{\delta,H}\right) \mathrm{d}s \,\middle|\, \mathcal{G}_t\right].$$

Approximation to the Optimal Strategy

Recall that

$$\pi_t^* = \left[\frac{\lambda(Z_t^{\delta,H})}{\gamma\sigma(Z_t^{\delta,H})} + \frac{\rho q \xi_t}{\gamma\sigma(Z_t^{\delta,H})} \right] X_t$$

and ξ_t is from the martingale rep. of $M_t = \widetilde{\mathbb{E}} \left[e^{\frac{1-\gamma}{2q\gamma} \int_0^T \lambda^2(Z_s^{\delta,H})\, \mathrm{d}s} \middle| \mathcal{G}_t \right]$.

Theorem (Fouque-Hu '17)

The optimal strategy π_t^* is approximated by

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$$:= \pi_t^{(0)} + \delta^H \pi_t^{(1)} + \mathcal{O}(\delta^{2H}).$$

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How Good is the Approximation?

Corollary

In the case of power utility $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\pi^{(0)} = \frac{\lambda(Z_t^{\delta,H})}{\gamma\sigma(Z_t^{\delta,H})} X_t$ generates the approximation of V_t^δ up to order δ^H (leading order + two correction terms of order δ^H), thus asymptotically optimal in $\mathcal{A}_t^\delta$.

- $H = \frac{1}{2}$, $Z_t^{\delta,H}$ becomes the Markovian OU process, both approximation coincides with results in [Fouque Sircar Zariphopoulou '13]. The corollary recovers [Fouque -Hu '16].
- Sketch of proofs: Apply Taylor expansion to $\lambda(z)$ at the point $Z_0^{\delta,H}$, and then control the moments $Z_t^{\delta,H} - Z_0^{\delta,H}$.

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Merton Problem under General Utility

Martingale Distortion Transformation is not available
→ Start with a given strategy $\pi^{(0)}$

- A first order approximation to $V^{\pi^{(0)},\delta}$ obtained by epsilon-martingale decomposition[3][4]
- Optimality of $\pi^{(0)}$ in a smaller class of controls of feedback form

Denote by $v^{(0)}(t,x,z)$ the value function at the Sharpe-ratio $\lambda(z)$, we define $\pi^{(0)}$ by

$$\pi^{(0)}(t,x,z) = -\frac{\lambda(z)}{\sigma(z)}\frac{v_x^{(0)}(t,x,z)}{v_{xx}^{(0)}(t,x,z)}$$

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[3] Fouque Papanicolaou Sircar '01
[4] Garnier Solna '15

Merton Problem under General Utility

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- A first order approximation to $V^{\pi^{(0)},\delta}$
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Epsilon-Martingale Decomposition

Find $Q_t^{\pi^{(0)},\delta}$ such that

- $Q_T^{\pi^{(0)},\delta} = V_T^{\pi^{(0)},\delta} = U(X_T^{\pi^{(0)}})$,
- $Q_t^{\pi^{(0)},\delta} = M_t^\delta + R_t^\delta$, where M_t^δ is a martingale and R_t^δ is of order δ^{2H}.

Then

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First order approximation to $V^{\pi^{(0)},\delta}$

Proposition

For fixed $t \in [0,T)$, $X_t^{\pi^{(0)}} = x$, and the observed value $Z_0^{\delta,H}$, the $\mathcal{F}_t$-measurable value process $V_t^{\pi^{(0)},\delta}$ is of the form

$$V_t^{\pi^{(0)},\delta} = Q_t^{\pi^{(0)},\delta}(X_t^{\pi^{(0)}}, Z_0^{\delta,H}) + \mathcal{O}(\delta^{2H}),$$

where $Q_t^{\pi^{(0)},\delta}(x,z)$ is given by:

$$\begin{aligned} Q_t^{\pi^{(0)},\delta}(x,z) =& v^{(0)}(t,x,z) + \lambda(z)\lambda'(z)D_1 v^{(0)}(t,x,z)\phi_t^\delta \\ &+ \delta^H \rho\lambda^2(z)\lambda'(z)D_1^2 v^{(0)}(t,x,z)\frac{(T-t)^{H+3/2}}{\Gamma(H+\frac{5}{2})}. \end{aligned}$$

- For power utility, $Q_t^{\pi^{(0)},\delta}$ coincides with the approximation of V_t^δ
- In the Markovian case $H = \frac{1}{2}$, recovers the results in [Fouque-Hu '16]

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Asymptotically Optimality of $\pi^{(0)}$

Theorem (Fouque-Hu '17)

The trading strategy $\pi^{(0)}(t,x,z) = -\frac{\lambda(z)}{\sigma(z)}\frac{v_x^{(0)}(t,x,z)}{v_{xx}^{(0)}(t,x,z)}$ *is asymptotically optimal in the following class:*

$$\widetilde{\mathcal{A}}_t^{\delta}[\widetilde{\pi}^0, \widetilde{\pi}^1, \alpha] := \left\{ \pi = \widetilde{\pi}^0 + \delta^{\alpha}\widetilde{\pi}^1 : \pi \in \mathcal{A}_t^{\delta}, \alpha > 0, 0 < \delta \leq 1 \right\}.$$

The proof uses the nice properties of the risk tolerance function $R(t,x,z) = -\frac{v_x^{(0)}(t,x,z)}{v_{xx}^{(0)}(t,x,z)}$ and the operator $D_1 = R\partial_x$

Merton Problem under Fast-Varying Fractional SV

Consider a ϵ-scaled stationary fOU process $Y_t^{\epsilon,H}$

$$Y_t^{\epsilon,H} = \epsilon^{-H} \int_{-\infty}^{t} e^{-\frac{a(t-s)}{\epsilon}} \, \mathrm{d}W_s^{(H)} = \int_{-\infty}^{t} \mathcal{K}^{\epsilon}(t-s) \, \mathrm{d}W_s^{Y}$$

together with the risky asset

$$\mathrm{d}S_t = S_t \left[\mu(Y_t^{\epsilon,H}) \, \mathrm{d}t + \sigma(Y_t^{\epsilon,H}) \, \mathrm{d}W_t \right], \quad \mathrm{d} \left\langle W, W^Y \right\rangle_t = \rho \, \mathrm{d}t,$$

For power utilities, we obtain:

- The value process V_t^{ϵ} and the corresponding optimal strategy π^*
- First order approximations to V_t^{ϵ} and π^*
- A strategy $\pi^{(0)}$ to generate this approximated value process

Using singular perturbation , but only valid for $H \in (\frac{1}{2}, 1)$

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Approximation to the Value Process V_t^ϵ

Theorem (Fouque-Hu '17)

For fixed $t \in [0,T)$, $X_t = x$ the value process V_t^ϵ takes the form

$$V_t^\epsilon = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2 (T-t)} + \frac{X_t^{1-\gamma}}{\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2 (T-t)} \phi_t^\epsilon$$
$$+ \epsilon^{1-H} \rho \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2 (T-t)} \widetilde{\lambda} \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{\langle \lambda\lambda' \rangle (T-t)^{H+\frac{1}{2}}}{a\Gamma(H+\frac{3}{2})}$$
$$+ o(\epsilon^{1-H}),$$

where ϕ_t^ϵ is the random component of order ϵ^{1-H}

$$\phi_t^\epsilon = \mathbb{E}\left[\frac{1}{2} \int_t^T \left(\lambda^2 (Y_s^{\epsilon,H}) - \overline{\lambda}^2 \right) \mathrm{d}s \middle| \mathcal{G}_t \right].$$

Optimal Portfolio

Theorem (Fouque-Hu '17)

The optimal strategy π_t^* *is approximated by*

$$\pi_t^* = \left[\frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})} + \epsilon^{1-H}\frac{\rho(1-\gamma)}{\gamma^2\sigma(Y_t^{\epsilon,H})}\frac{\langle\lambda\lambda'\rangle\,(T-t)^{H-1/2}}{a\Gamma(H+\frac{1}{2})}\right]X_t$$
$$+\,o(\epsilon^{1-H})$$

Corollary

In the case of power utility, $\pi^{(0)} = \frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})}X_t$ generates the approximation of V^ϵ up to order ϵ^{1-H} (leading order + two correction terms of order ϵ^{1-H}), thus asymptotically optimal in $\mathcal{A}_t^\epsilon$.

Comparison with the Markovian Case

- The value function and the optimal strategy are derived in [FSZ '13]:

$$V^{\epsilon}(t, X_t) = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)} \left[1 - \sqrt{\epsilon}\rho\left(\frac{1-\gamma}{\gamma}\right)^2 \frac{\langle \lambda\theta' \rangle}{2}(T-t)\right] + \mathcal{O}(\epsilon)$$

$$\pi^*(t, X_t, Y_t^{\epsilon,H}) = \left[\frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})} + \sqrt{\epsilon}\frac{\rho(1-\gamma)}{\gamma^2\sigma(Y_t^{\epsilon,H})}\frac{\theta'(Y_t^{\epsilon,H})}{2}\right] X_t + \mathcal{O}(\epsilon)$$

- Formally let $H \downarrow \frac{1}{2}$ in our results:

$$V_t^{\epsilon} = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)} \left[1 + \sqrt{\epsilon}\rho\left(\frac{1-\gamma}{\gamma}\right)^2 \frac{\widetilde{\lambda}\langle \lambda\lambda' \rangle}{a}(T-t)\right] + o(\sqrt{\epsilon})$$

$$\pi_t^* = \left[\frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})} + \sqrt{\epsilon}\frac{\rho(1-\gamma)}{\gamma^2\sigma(Y_t^{\epsilon,H})}\frac{\langle \lambda\lambda' \rangle}{a}\right] X_t + o(\sqrt{\epsilon})$$

Joyeux Anniversaire Jim!